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ON THE STEADY VIBRATIONS OF THE THERMOELASTIC POROUS MATERIALS

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Abstract—The problem of the steady vibrations in a linear theory of homogeneous and isotropic thermoelastic solids with voids is considered. First, integral relations of Betti type are established. The singular solutions corresponding to concentrated sources are used to derive representations of Somigliana type. Then, radiation conditions are introduced and a uniqueness result for the exterior problem is established. The potentials of single layer and double layer are used to reduce the boundary value problems to singular integral equations for which Fredholm's theorems hold.

1. INTRODUCTION

The theory of elastic materials with voids is one of the simple extensions of the classical theory of elasticity for the treatment of porous solids in which the matrix material is elastic and the interstices are void of material. The theory of elastic materials with voids seems to be an adequate tool to describe the behaviour of granular materials like rock, soils and manufactured porous bodies. The non-linear theory of elastic materials with voids has been established by Nunziato and Cowin (1979). In this theory the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. This representation introduces an additional degree of kinematic freedom. The linear theory of elastic materials with voids has been established by Cowin and Nunziato (1983). An extension of this theory to thermoelastic bodies was proposed by Iesan (1986). The theory of thermoelastic materials with voids has been studied in various papers (see e.g. Iesan, 1987; Ciarletta and Scarpetta, 1989; Chandrasekharaiah and Cowin, 1989; Ciarletta and Scalia, 1990; Ciarletta, 1991; Ciarletta and Scalia, 1993). In this paper we study the problem of steady vibrations in the theory of thermoelastic materials with voids by using the method of potentials developed by Kupradze (1965) and Kupradze *et al.* (1979). The wave propagation in the isothermal theory of elastic materials with voids has been studied in various papers (see e.g. Cowin and Nunziato, 1983; Nunziato and Walsh, 1977; Puri and Cowin, 1985; Chandrasekharaiah, 1987).

In Section 2 we present the basic equations of the dynamic theory of thermoelastic materials with voids governing the steady vibrations. Within the framework of the linear theory of thermoelastic materials with voids, Ciarletta (1991), by extending a result of classical thermoelasticity obtained by Nowacki (1964), has established a solution of Galerkin type and derived the solutions for steady vibrations corresponding to a concentrated heat source and a concentrated extrinsic body force. In Section 3 we use the solution of Galerkin type to obtain the singular solutions of the field equations corresponding to concentrated classical body forces and introduce the matrix of fundamental solutions in the case of steady vibrations. In Section 4 we derive some useful integral identities of Betti type. Section 5 is devoted to a representation of the regular solutions. In Section 6 we present the radiation conditions and establish a uniqueness theorem for the exterior problem. The fundamental solutions are used in Section 7 to derive representations of Somigliana type and to introduce potentials of single and double layers. The boundary value problems are reduced to singular integral equations for which Fredholm's basic theorems are valid.

2. PRELIMINARIES AND BASIC EQUATIONS

Throughout this paper we employ a rectangular Cartesian system Ox_i ($i = 1, 2, 3$). We shall employ the usual summation and differentiation conventions: Latin subscripts (unless otherwise specified) are understood to range over the integers (1, 2, 3), summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

Let u_i denote the components of the displacement vector field. Then the components of the infinitesimal strain field are given by

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{1}$$

We consider thermoelastic materials with voids which possess a reference configuration in which the volume fraction and the absolute temperature T_0 (>0) are constants. We denote by φ the change in volume fraction from the reference volume fraction and by ϑ the temperature measured from the absolute temperature T_0. The constitutive equations for homogeneous and isotropic thermoelastic bodies are (Iesan, 1987)

$$\begin{aligned} &t_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + \beta\varphi\delta_{ij} - b\vartheta\delta_{ij}, \quad h_i = \alpha\varphi_{,i}, \\ &g = -\beta e_{rr} - \xi\varphi + m\vartheta, \quad \rho\eta = be_{rr} + a\vartheta + m\varphi, \quad q_i = k\vartheta_{,i}. \end{aligned} \tag{2}$$

Here t_{ij} are the components of the stress tensor, h_i are the components of the equilibrated stress vector, g is the intrinsic equilibrated body force, η is the specific entropy, q_i are the components of the heat flux vector, ρ is the density in the reference configuration, λ, μ, β, b, α, ξ, m, a and k are constitutive constants, and δ_{ij} is Kronecker's delta.

The equations of motion can be written in the form

$$t_{ji,j} + \rho f_i = \rho\ddot{u}_i, \quad h_{i,i} + g + \rho l = \rho\kappa\ddot{\varphi}. \tag{3}$$

The energy equation is given by

$$\rho T_0\dot{\eta} = q_{i,i} + \rho s. \tag{4}$$

In eqns (3) and (4) we have used the following notations: f_i are the components of the body force vector, l is the extrinsic equilibrated body force, s is the extrinsic heat supply and κ is the equilibrated inertia. The surface traction $\mathbf{t}$, the equilibrated stress h and the heat flux q acting at a point $x(x_k)$ on the surface Σ are given by

$$t_i = t_{ji}n_j, \quad h = h_i n_i, \quad q = q_i n_i, \tag{5}$$

where $n_j = \cos(n_x, x_j)$ and n_x is the unit vector of the outward normal to Σ at x. From eqns (1), (2), (3) and (4) we obtain the field equations in terms of the displacement, volume fraction and temperature

$$\begin{aligned} &\mu\Delta u_i + (\lambda+\mu)u_{j,ji} + \beta\varphi_{,i} - b\vartheta_{,i} + \rho f_i = \rho\ddot{u}_i, \\ &\alpha\Delta\varphi - \xi\varphi - \beta u_{i,i} + m\vartheta + \rho l = \rho\kappa\ddot{\varphi}, \quad k\Delta\vartheta - bT_0\dot{u}_{i,i} - aT_0\dot{\vartheta} - mT_0\dot{\varphi} = -\rho s, \end{aligned} \tag{6}$$

where Δ is the Laplacian.

In what follows we study the problem of steady vibrations. We assume that

$$\begin{aligned} f_j &= \mathrm{Re}\,[f_j'(x)\exp(-\mathrm{i}\omega t)], & l &= \mathrm{Re}\,[l'(x)\exp(-\mathrm{i}\omega t)], \\ s &= \mathrm{Re}\,[s'(x)\exp(-\mathrm{i}\omega t)], & u_j &= \mathrm{Re}\,[u_j'(x;\omega)\exp(-\mathrm{i}\omega t)], \\ \varphi &= \mathrm{Re}\,[\varphi'(x;\omega)\exp(-\mathrm{i}\omega t)], & \vartheta &= \mathrm{Re}\,[\vartheta'(x;\omega)\exp(-\mathrm{i}\omega t)], \end{aligned}$$

where $\mathrm{i} = \sqrt{-1}$, Re $[f]$ denotes the real part of f and ω is a prescribed frequency. Clearly, we can write

$$e_{rs} = \mathrm{Re}\,[e'_{rs}(x;\omega)\exp(-\mathrm{i}\omega t)], \quad t_{rs} = \mathrm{Re}\,[t'_{rs}(x;\omega)\exp(-\mathrm{i}\omega t)],$$
$$h_j = \mathrm{Re}\,[h'_j(x;\omega)\exp(-\mathrm{i}\omega t)], \quad g = \mathrm{Re}\,[g'(x;\omega)\exp(-\mathrm{i}\omega t)],$$
$$\eta = \mathrm{Re}\,[\eta'(\mathrm{x};\omega)\exp(-\mathrm{i}\omega t)], \quad q_j = \mathrm{Re}\,[q'_j(x;\omega)\exp(-\mathrm{i}\omega t)],$$

where

$$e'_{ij} = \tfrac{1}{2}(u'_{i,j}+u'_{j,i}),$$

and

$$t'_{ij} = \lambda e'_{rr}\delta_{ij}+2\mu e'_{ij}+\beta\varphi'\delta_{ij}-b\vartheta'\delta_{ij}, \quad h'_i = \alpha\varphi'_{,i},$$
$$g' = -\beta e'_{rr}-\xi\varphi'+m\vartheta', \quad \rho\eta' = be'_{rr}+a\vartheta'+m\varphi', \quad q'_i = k\vartheta'_{,i}. \tag{7}$$

We assume that $\lambda+2\mu$, μ, ρ, α and κ are positive. We introduce the notations

$$\mu' = \frac{\mu}{\rho}, \quad \lambda' = \frac{\lambda}{\rho}, \quad \beta' = \frac{\beta}{\rho}, \quad b' = \frac{b}{\rho}, \quad F_i = -f_i, \quad \xi' = -\frac{\xi}{\alpha},$$
$$\gamma = \frac{\beta}{\alpha}, \quad m' = \frac{m}{\alpha}, \quad L = -\frac{\rho l}{\alpha}, \quad \chi = -\xi'+\eta\omega^2, \quad \eta = \rho\frac{\kappa}{\alpha}, \quad T'_0 = \rho\frac{T_0}{k},$$
$$S = -\frac{\rho}{k}s, \quad a' = \frac{a}{\rho}, \quad c = \frac{\alpha m'}{\rho} = \frac{m}{\rho}, \quad t'_{ij} = \rho T_{ij}, \quad h'_j = \alpha H_j, \quad q'_j = kQ_j. \tag{8}$$

From eqn (6) we obtain the basic set of differential equations of steady vibrations. In what follows, for convenience, we suppress the primes so that the equations take the form

$$\mu\Delta u_r+(\lambda+\mu)u_{j,jr}+\beta\varphi_{,r}-b\vartheta_{,r}+\omega^2 u_r = F_r,$$
$$\Delta\varphi-\gamma u_{r,r}+\chi\varphi+m\vartheta = L, \quad \Delta\vartheta+\mathrm{i}\omega T_0 bu_{r,r}+\mathrm{i}\omega T_0 c\varphi+\mathrm{i}\omega T_0 a\vartheta = S. \tag{9}$$

The system (9) can be written in the matrix form. As in Kupradze (1965) the vector $v = (v_1, v_2, \ldots, v_m)$ shall be considered as the column matrix

$$v = \begin{Vmatrix} v_1 \\ \vdots \\ v_m \end{Vmatrix}.$$

Thus, the product of the matrix $A = \|a_{ij}\|_{m\times m}$ and the vector $v = (v_1, v_2, \ldots, v_m)$ is an m-dimensional vector. The vector v multiplied by the matrix A will denote the matrix product between the row matrix $\|v_1, v_2, \ldots, v_m\|$ and the matrix A. We introduce the matricial differential operator

$$A\left(\frac{\partial}{\partial x}, \omega\right) = \left\| A_{rs}\left(\frac{\partial}{\partial x}, \omega\right)\right\|_{5\times 5}, \tag{10}$$

where

$$A_{rs}\left(\frac{\partial}{\partial x},\omega\right) = \mu\delta_{rs}\Delta+(\lambda+\mu)\frac{\partial^2}{\partial x_r\,\partial x_s}+\omega^2\delta_{rs},\quad A_{r4} = \beta\frac{\partial}{\partial x_r},$$

$$A_{r5} = -b\frac{\partial}{\partial x_r},\quad A_{4r} = -\gamma\frac{\partial}{\partial x_r},\quad A_{44} = \Delta+\chi,\quad A_{45} = m,$$

$$A_{5r} = \mathrm{i}\omega b T_0\frac{\partial}{\partial x_r},\quad A_{54} = \mathrm{i}\omega T_0 c,\quad A_{55} = \Delta+\mathrm{i}\omega T_0 a. \tag{11}$$

If we denote

$$U = (u_1, u_2, u_3, \varphi, \vartheta),\quad F = (F_1, F_2, F_3, L, S) \tag{12}$$

then we can write the equation (9) in the form

$$A\left(\frac{\partial}{\partial x},\omega\right)U = F. \tag{13}$$

Clearly, we may write

$$A\left(\frac{\partial}{\partial x},\omega\right) = A_0(\omega)+A_1\left(\frac{\partial}{\partial x},\omega\right)+A_2\left(\frac{\partial}{\partial x}\right),$$

where

$$A_0(\omega) = \|a_{rs}\|_{5\times 5},\quad A_1\left(\frac{\partial}{\partial x},\omega\right) = \left\|B_{rs}\left(\frac{\partial}{\partial x},\omega\right)\right\|_{5\times 5},\quad A_2\left(\frac{\partial}{\partial x}\right) = \left\|\Lambda_{rs}\left(\frac{\partial}{\partial x}\right)\right\|_{5\times 5},$$

$$a_{rs}(\omega) = \omega^2\delta_{rs},\quad a_{i4} = a_{4i} = a_{i5} = a_{5i} = 0,\quad a_{44} = \chi,$$

$$a_{45} = m,\quad a_{54} = \mathrm{i}\omega T_0 c,\quad a_{55} = \mathrm{i}\omega T_0 a,\quad B_{44} = B_{55} = B_{54} = B_{45} = 0,$$

$$B_{rs}\left(\frac{\partial}{\partial x},\omega\right) = 0,\quad B_{4i} = -\gamma\frac{\partial}{\partial x_i},\quad B_{5i} = \mathrm{i}\omega T_0 b\frac{\partial}{\partial x_i},\quad B_{i4} = \beta\frac{\partial}{\partial x_i},$$

$$B_{i5} = -b\frac{\partial}{\partial x_i},\quad \Lambda_{rs}\left(\frac{\partial}{\partial x}\right) = \mu\delta_{rs}\Delta+(\lambda+\mu)\frac{\partial^2}{\partial x_r\,\partial x_s},$$

$$\Lambda_{44} = \Lambda_{55} = \Delta,\quad \Lambda_{4i} = \Lambda_{5i} = \Lambda_{i4} = \Lambda_{i5} = \Lambda_{45} = \Lambda_{54} = 0. \tag{14}$$

For any real unit vector $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$ we have

$$\det\|\Lambda_{rs}(\zeta)\|_{5\times 5} = (\lambda+2\mu)\mu^2. \tag{15}$$

Since $\lambda+2\mu > 0$ it follows from eqn (15) that the system (9) is elliptic.

Let $\tilde{A}$ be the adjoint operator of the operator A. Clearly,

$$\tilde{A}\left(\frac{\partial}{\partial x},\omega\right) = A_0^*(\omega)-A_1^*\left(\frac{\partial}{\partial x},\omega\right)+A_2\left(\frac{\partial}{\partial x}\right),$$

where A^* is the transposed matrix of A.

The adjoint system is

$$\tilde{A}\left(\frac{\partial}{\partial x},\omega\right)V = \Phi, \tag{16}$$

where $V = (v_1, v_2, v_3, v_4, v_5)$ and $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)$.

The system (16) may be written in the form

$$\begin{aligned}\mu\Delta v_r+(\lambda+\mu)v_{j,jr}+\gamma v_{4,r}-\mathrm{i}\omega T_0 v_{5,r}+\omega^2 v_r &= \phi_r,\\ \Delta v_4-\beta v_{r,r}+\chi v_4+\mathrm{i}\omega T_0 c v_5 &= \phi_4,\\ \Delta v_5+b v_{r,r}+m v_4+\mathrm{i}\omega T_0 a v_5 &= \phi_5.\end{aligned} \tag{17}$$

3. FUNDAMENTAL SOLUTIONS

In this section we present the basic solutions corresponding to a concentrated body force, a concentrated extrinsic equilibrated body force and a concentrated heat source in the case of steady vibrations. To this end we shall use the results established by Ciarletta (1991).

If we introduce the notations

$$\begin{aligned}\Box_1 &= (\lambda+2\mu)\Delta+\omega^2, & \Box_2 &= \mu\Delta+\omega^2,\\ \Box_3 &= \Delta+\chi, & \Box_4 &= \Delta+\mathrm{i}\omega T_0 a\end{aligned} \tag{18}$$

then, eqns (9) become

$$\begin{aligned}&\Box_2 u_r+(\lambda+\mu)u_{j,jr}+\beta\varphi_{,r}-b\vartheta_{,r} = F_r,\\ &\Box_3\varphi-\gamma u_{r,r}+m\vartheta = L, \quad \Box_4\vartheta+\mathrm{i}\omega T_0 b u_{r,r}+\mathrm{i}\omega T_0 c\varphi = S.\end{aligned} \tag{19}$$

Let us introduce the notation

$$\Upsilon = \Box_1\Box_2\Box_3+\gamma\beta\Delta\Box_4+\mathrm{i}\omega T_0[b^2\Delta\Box_3+b(c\gamma+\beta m)\Delta-mc\Box_1].$$

By using a representation of Galerkin type, Ciarletta (1991) has established the solutions corresponding to a concentrated heat source and a concentrated extrinsic body force acting in a body occupying the entire space. We have used the same representation and have derived the solutions corresponding to concentrated classical body forces. Thus, if we assume that $F_i = \delta(x-y)\delta_{ij}$, $L = 0$, $S = 0$, where δ is the Dirac delta and j is fixed, then we find that

$$\begin{aligned}u_k^{(j)} &= \delta_{kj}\Upsilon E-\{(\lambda+\mu)\Box_3\Box_4+\beta\gamma\Box_4+\mathrm{i}\omega T_0[mb\beta+bc\gamma+b^2\Box_3-mc(\lambda+\mu)]\}E_{,kj},\\ \varphi^{(j)} &= (\mathrm{i}\omega T_0 mb+\gamma\Box_4)\Box_2 E_{,j}, \quad \vartheta^{(j)} = -\mathrm{i}\omega T_0(c\gamma+b\Box_3)\Box_2 E_{,j}, \quad (j=1,2,3),\end{aligned} \tag{20}$$

where E is given by

$$E(x;y) = -\frac{1}{4\pi\mu(\lambda+2\mu)r}\sum_{j=1}^{4} a_j \exp(\mathrm{i}\sigma_j r). \tag{21}$$

Here $r = |x-y|$ and σ_j are the roots with positive real parts of the following equation

$$\begin{aligned}&(\lambda+2\mu)\sigma^6+[(\mathrm{i}\omega T_0 a+\chi)(\lambda+2\mu)+\omega^2+\gamma\beta+\mathrm{i}\omega T_0 b^2]\sigma^4\\ &\quad+\{\mathrm{i}\omega T_0[(\lambda+2\mu)(a\chi-mc)+\omega^2 a+a\gamma\beta+b^2\chi+b(c\gamma+\beta m)]+\omega^2\chi\}\sigma^2\\ &\quad+\mathrm{i}\omega^3 T_0(a\chi-mc) = 0,\end{aligned} \tag{22}$$

and $\sigma_4 = \omega/\sqrt{\mu}$. Moreover,

$$a_s^{-1} = \prod_{j=1(j\neq s)}^{4} (\sigma_s^2-\sigma_j^2), \quad (s = 1,2,3,4). \tag{23}$$

In the case of the loadings $F_i = 0$, $L = \delta(x-y)$ and $S = 0$ we obtain that the solution is (Ciarletta, 1991)

$$u_j^{(4)} = -(\mathrm{i}\omega T_0 cb+\beta\Box_4)P_{,j},$$
$$\varphi^{(4)} = (\mathrm{i}\omega T_0 b^2\Delta+\Box_1\Box_4)P, \quad \vartheta^{(4)} = \mathrm{i}\omega T_0(b\beta\Delta-c\Box_1)P, \tag{24}$$

where P is given by

$$P(x;y) = \frac{-1}{4\pi(\lambda+2\mu)r}\sum_{j=1}^{3} b_j \exp(\mathrm{i}\sigma_j r),$$
$$b_1^{-1} = (\sigma_1^2-\sigma_2^2)(\sigma_1^2-\sigma_3^2), \quad b_2^{-1} = (\sigma_2^2-\sigma_1^2)(\sigma_2^2-\sigma_3^2),$$
$$b_3^{-1} = (\sigma_3^2-\sigma_1^2)(\sigma_3^2-\sigma_2^2). \tag{25}$$

Finally, the solution corresponding to $F_i = 0$, $L = 0$ and $S = \delta(x-y)$ is (Ciarletta, 1991)

$$u_j^{(5)} = (m\beta+b\Box_3)P_{,j}, \quad \varphi^{(5)} = (b\gamma\Delta-m\Box_1)P, \quad \vartheta^{(5)} = (\beta\gamma\Delta+\Box_1\Box_3)P. \tag{26}$$

The functions $u_j^{(s)}$, $\varphi^{(s)}$, $\vartheta^{(s)}$ $(s = 1,2,\ldots,5)$ given by eqns (20), (24) and (26) represent the fundamental solutions in the case of steady vibrations.

Let $\Gamma(x,y;\omega)$ be the matrix of fundamental solutions of the system (9)

$$\Gamma(x,y;\omega) = \|\Gamma_{rs}(x,y;\omega)\|_{5\times 5}, \tag{27}$$

where

$$\Gamma_{jk}(x,y;\omega) = u_j^{(k)}, \quad \Gamma_{4k}(x,y;\omega) = \varphi^{(k)}(x,y),$$
$$\Gamma_{5k}(x,y;\omega) = \vartheta^{(k)}(x,y), \quad k = 1,2,\ldots,5.$$

If $x \neq y$ each column $\Gamma^{(s)}$ $(s = 1,2,\ldots,5)$ of the matrix $\Gamma(x,y;\omega)$ satisfies at x the homogeneous system (9). Thus,

$$A\left(\frac{\partial}{\partial x},\omega\right)\Gamma(x,y;\omega) = 0, \quad x \neq y. \tag{28}$$

Let $\tilde{\Gamma}(x,y;\omega)$ be the matrix of fundamental solutions of the system (17),

$$\tilde{A}\left(\frac{\partial}{\partial x},\omega\right)\tilde{\Gamma}(x,y;\omega) = 0, \quad x \neq y. \tag{29}$$

It is easy to prove that

$$\tilde{\Gamma}^*(y,x;\omega) = \Gamma(x,y;\omega). \tag{30}$$

4. RECIPROCITY RELATIONS

Let D^+ be a finite region bounded by the closed Liapunov surface $\mathscr{F}$, and D^- the complementary of $D^+ \cup \mathscr{F}$ to infinite space. In what follows letters in boldface stand for three-dimensional vectors. We say that $U = (u_1,u_2,u_3,u_4,u_5) = (\mathbf{u},u_4,u_5)$ is regular in D^+

if $u_s \in C^1(\bar{D}^+) \cap C^2(D^+)$ and $u_{s,rm}$ $(s = 1, 2, \ldots, 5)$ are integrable on D^+. Let $\Sigma(x;\delta)$ be the sphere with the centre in x and radius δ. We say that $U = (\mathbf{u}, u_4, u_5)$ is regular in D^- if $u_s \in C^1(\bar{D}^-) \cap C^2(\bar{D})$ and $u_{s,rm}$ $(s = 1, 2, \ldots, 5)$ are integrable on $D^- \cap \Sigma(0, \delta)$ for any δ.

Let $A_i(\partial/\partial x, \omega)$ be the row matrix with the elements $A_{ij}(\partial/\partial x, \omega)$ $(i, j = 1, 2, \ldots, 5)$ given by eqn (11). If we introduce the notations

$$L_r U = A_r\left(\frac{\partial}{\partial x_r}, \omega\right) U, \quad (r = 1, 2, \ldots, 5),$$

then the system (9) can be written in the form

$$L_i U = F_i, \quad L_4 U = L, \quad L_5 U = S. \tag{31}$$

Let $\tilde{A}_i(\partial/\partial x, \omega)$ be the row-matrix with the elements $\tilde{A}_{ij}(\partial/\partial x, \omega)$ $(i, j = 1, 2, \ldots, 5)$. If we denote

$$\tilde{L}_r V = \tilde{A}_r\left(\frac{\partial}{\partial x_r}, \omega\right) V, \quad (r = 1, 2, \ldots, 5),$$

where $V = (\mathbf{v}, v_4, v_5)$, then the system (17) becomes

$$\tilde{L}_i V = \phi_i, \quad \tilde{L}_4 V = \phi_4, \quad \tilde{L}_5 V = \phi_5. \tag{32}$$

Let $U = (\mathbf{u}, u_4, u_5)$ and $V = (\mathbf{v}, v_4, v_5)$ be regular vectors in D^+ which satisfy eqns (13) and (16), respectively. We introduce the notations

$$\begin{aligned}
&T_{ij} = \lambda u_{r,r}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) + \beta u_4 \delta_{ij} - b u_5 \delta_{ij}, \quad H_i = u_{4,i},\\
&G = \chi u_4 - \gamma u_{r,r} + m u_5, \quad Q_i = u_{5,i}, \quad M = \mathrm{i}\omega T_0(b u_{r,r} + c u_4 + a u_5),\\
&\tilde{T}_{ij} = \lambda v_{r,r}\delta_{ij} + \mu(v_{i,j} + v_{j,i}) + \gamma v_4 \delta_{ij} - \mathrm{i}\omega T_0 b v_5 \delta_{ij}, \quad \tilde{H}_i = v_{4,i},\\
&\tilde{G} = \chi v_4 - \beta v_{r,r} + \mathrm{i}\omega T_0 c v_5, \quad \tilde{Q}_i = v_{5,i}, \quad \tilde{M} = b v_{r,r} + m v_4 + \mathrm{i}\omega T_0 a v_5.
\end{aligned} \tag{33}$$

Then, eqns (31) and (32) may be written in the forms

$$T_{ji,j} + \omega^2 u_i = F_i, \quad H_{i,i} + G = L, \quad Q_{i,i} + M = S, \tag{34}$$

and

$$\tilde{T}_{ji,j} + \omega^2 v_i = \phi_i, \quad \tilde{H}_{i,i} + \tilde{G} = \phi_4, \quad \tilde{Q}_{i,i} + \tilde{M} = \phi_5, \tag{35}$$

respectively. It follows from eqns (31) and (34) that

$$\begin{aligned}
v_i L_i U + v_4 L_4 U + v_5 L_5 U &= v_i(T_{ji,j} + \omega^2 u_i) + v_4(H_{i,i} + G) + v_5(Q_{i,i} + M)\\
&= (v_i T_{ji})_{,j} + (v_4 H_i)_{,i} + (v_5 Q_i)_{,i} - v_{i,j}T_{ji} - v_{4,i}H_i - v_{5,i}Q_i + v_4 G + v_5 M + \omega^2 u_i v_i.
\end{aligned}$$

Applying the divergence theorem, we are led to

$$\begin{aligned}
\int_{D^+} VA\left(\frac{\partial}{\partial x}, \omega\right) U \,\mathrm{d}v = \int_{\mathscr{S}} \left(v_i T_{ji} n_j + v_4 \frac{\partial u_4}{\partial n} + v_5 \frac{\partial u_5}{\partial n}\right) \mathrm{d}a - \int_{D^+} [E(\mathbf{u}, \mathbf{v})\\
+ \beta u_4 v_{r,r} + u_{4,i} v_{4,i} - \chi u_4 v_4 - \mathrm{i}\omega T_0 c u_4 v_5 - b u_5 v_{r,r} + u_{5,i} v_{5,i}\\
- m u_5 v_4 - \mathrm{i}\omega T_0 a u_5 v_5 + \gamma u_{r,r} v_4 - \mathrm{i}\omega T_0 b u_{r,r} v_5 - \omega^2 u_i v_i] \,\mathrm{d}v,
\end{aligned} \tag{36}$$

where

$$\frac{\partial f}{\partial n} = f_{,i}n_i \quad \text{and} \quad E(\mathbf{u},\mathbf{v}) = \lambda u_{r,r}v_{j,j} + \frac{\mu}{2}(u_{i,j}+u_{j,i})(v_{i,j}+v_{j,i}). \tag{37}$$

In a similar manner we find that

$$\int_{D^+} U\tilde{A}\left(\frac{\partial}{\partial x},\omega\right)V\,\mathrm{d}v = \int_{\mathscr{F}}\left(u_i\tilde{T}_{ji}n_j + u_4\frac{\partial v_4}{\partial n} + u_5\frac{\partial v_5}{\partial n}\right)\mathrm{d}a - \int_{D^+}[E(\mathbf{u},\mathbf{v})$$
$$+\beta u_4 v_{r,r} + u_{4,i}v_{4,i} - \chi u_4 v_4 - \mathrm{i}\omega T_0 c u_4 v_5 - b u_5 v_{r,r}$$
$$+u_{5,i}v_{5,i} - m u_5 v_4 - \omega T_0 a u_5 v_5 + \gamma u_{r,r}v_4 - \mathrm{i}\omega T_0 b u_{r,r} v_5 - \omega^2 u_i v_i]\,\mathrm{d}v. \tag{38}$$

Thus, from eqns (36) and (38) we conclude that

$$\int_{D^+}\left[U\tilde{A}\left(\frac{\partial}{\partial x},\omega\right)V - VA\left(\frac{\partial}{\partial x},\omega\right)U\right]\mathrm{d}v = \int_{\mathscr{F}}\left[U\tilde{R}\left(\frac{\partial}{\partial x},n_x\right)V - VR\left(\frac{\partial}{\partial x},n_x\right)U\right]\mathrm{d}a, \tag{39}$$

where

$$R\left(\frac{\partial}{\partial x},N_x\right) = \left\|R_{rs}\left(\frac{\partial}{\partial x},n_x\right)\right\|_{5\times 5}, \quad \tilde{R}\left(\frac{\partial}{\partial x},n_x\right) = \left\|\tilde{R}_{rs}\left(\frac{\partial}{\partial x},n_x\right)\right\|_{5\times 5},$$

$$R_{ij}\left(\frac{\partial}{\partial x},n_x\right) = \mu\delta_{ij}\frac{\partial}{\partial n} + \lambda n_i\frac{\partial}{\partial x_j} + \mu n_j\frac{\partial}{\partial x_i}, \quad R_{i4} = \beta n_i, \quad R_{i5} = -bn_i,$$

$$R_{4i} = R_{5i} = R_{45} = R_{54} = 0, \quad \tilde{R}_{ij}\left(\frac{\partial}{\partial x},n_x\right) = R_{ij}\left(\frac{\partial}{\partial x},n_x\right), \quad \tilde{R}_{i4} = \gamma n_i,$$

$$\tilde{R}_{4i} = \tilde{R}_{5i} = \tilde{R}_{45} = \tilde{R}_{54} = 0, \quad \tilde{R}_{i5} = -\mathrm{i}\omega T_0 b n_i, \quad R_{44} = R_{55} = \tilde{R}_{44} = \tilde{R}_{55} = \frac{\partial}{\partial n}. \tag{40}$$

Let $R_i(\partial/\partial x, n_x)$ be row matrix with the elements $R_{ij}(\partial/\partial x, n_x)$, $(i,j = 1,2,\ldots,5)$. Then, from eqns (33) and (40) we obtain

$$T_{ij}n_j = R_i\left(\frac{\partial}{\partial x},n_x\right)U, \quad H_i n_i = R_4\left(\frac{\partial}{\partial x},n_x\right)U, \quad Q_i n_i = R_5\left(\frac{\partial}{\partial x},n_x\right)U. \tag{41}$$

Next, we establish other integral identities. In what follows we denote by $\bar{f}$ the complex conjugate of f. We assume that $U = (\mathbf{u}, u_4, u_5)$ is a regular vector field in D^+ that satisfies the equation

$$A\left(\frac{\partial}{\partial x},\omega\right)U = 0. \tag{42}$$

Then, the vector $\bar{U} = (\bar{\mathbf{u}}, \bar{u}_4, \bar{u}_5)$ satisfies the equations

$$\mu\Delta\bar{u}_r + (\lambda+\mu)\bar{u}_{j,jr} + \beta\bar{u}_{4,r} - b\bar{u}_{5,r} + \omega^2\bar{u}_r = 0,$$
$$\Delta\bar{u}_4 - \gamma\bar{u}_{r,r} + \chi\bar{u}_4 + m\bar{u}_5 = 0,$$
$$\Delta\bar{u}_5 - \mathrm{i}\omega T_0 b\bar{u}_{r,r} - \mathrm{i}\omega T_0 c\bar{u}_4 - \mathrm{i}\omega T_0 a\bar{u}_5 = 0. \tag{43}$$

These equations can be written in the form

$$\tilde{T}_{ij,j}+\omega^2\tilde{u}_i = 0, \quad \tilde{H}_{i,i}+\tilde{G} = 0, \quad \tilde{Q}_{i,i}+\tilde{M} = 0, \tag{44}$$

where

$$\tilde{T}_{ij} = \lambda\tilde{u}_{r,r}\delta_{ij}+\mu(\tilde{u}_{i,j}+\tilde{u}_{j,i})+\beta\tilde{u}_4\delta_{ij}-b\tilde{u}_5\delta_{ij}, \quad \tilde{H}_i = \tilde{u}_{4,i},$$
$$\tilde{G} = \chi\tilde{u}_4-\gamma\tilde{u}_{r,r}+m\tilde{u}_5, \quad \tilde{Q}_i = \tilde{u}_{5,i}, \quad \tilde{M} = -\mathrm{i}\omega T_0(b\tilde{u}_{r,r}+c\tilde{u}_4+a\tilde{u}_5).$$

In the case of a homogeneous problem, in view of eqn (34), we obtain

$$\int_{D^+}\left[(T_{ji,j}+\omega^2u_i)\tilde{u}_i+\frac{\beta}{\gamma}(H_{i,i}+G)\tilde{u}_4+\frac{1}{\mathrm{i}\omega T_0}(Q_{i,i}+M)\tilde{u}_5\right]\mathrm{d}v = 0. \tag{45}$$

Taking into account that from eqn (8) we have $\gamma c = m\beta$ and with the help of divergence theorem, by eqn (45), we get

$$\int_{\mathcal{F}}(T_{ji}n_j\tilde{u}_i+\frac{\beta}{\gamma}H_in_i\tilde{u}_4+\frac{1}{\mathrm{i}\omega T_0}Q_in_i\tilde{u}_5)\,\mathrm{d}a+\int_{D^+}\Big[-E(\mathbf{u},\tilde{\mathbf{u}})$$
$$-\beta(u_4\tilde{u}_{j,j}+\tilde{u}_4u_{i,i})+b(u_5\tilde{u}_{s,s}+\tilde{u}_5u_{r,r})+c(u_5\tilde{u}_4+\tilde{u}_5u_4)$$
$$-\frac{\beta}{\gamma}u_{4,i}\tilde{u}_{4,i}+\omega^2u_i\tilde{u}_i+\frac{\beta}{\gamma}\chi u_4\tilde{u}_4+au_5\tilde{u}_5-\frac{1}{\mathrm{i}\omega T_0}u_{5,i}\tilde{u}_{5,i}\Big]\mathrm{d}v = 0. \tag{46}$$

In a similar manner we find that

$$\int_{\mathcal{F}}(\tilde{T}_{ji}n_j\tilde{u}_i+\frac{\beta}{\gamma}\tilde{H}_in_iu_4-\frac{1}{\mathrm{i}\omega T_0}\tilde{Q}_in_iu_5)\,\mathrm{d}a+\int_{D^+}\Big[-E(\mathbf{u},\tilde{\mathbf{u}})$$
$$-\beta(\tilde{u}_4u_{j,j}+u_4\tilde{u}_{i,i})+b(\tilde{u}_5u_{s,s}+u_5\tilde{u}_{r,r})+c(\tilde{u}_5u_4+u_5\tilde{u}_4)$$
$$-\frac{\beta}{\gamma}u_{4,i}\tilde{u}_{4,i}+\omega^2u_i\tilde{u}_i+\frac{\beta}{\gamma}\chi u_4\tilde{u}_4+au_5\tilde{u}_5+\frac{1}{\mathrm{i}\omega T_0}u_{5,i}\tilde{u}_{5,i}\Big]\mathrm{d}v = 0. \tag{47}$$

It follows from eqns (46) and (47) that

$$\int_{\mathcal{F}}(T_{ji}n_j\tilde{u}_i+\frac{\beta}{\gamma}H_in_i\tilde{u}_4+\frac{1}{\mathrm{i}\omega T_0}Q_in_i\tilde{u}_5)\,\mathrm{d}a-\int_{\mathcal{F}}(\tilde{T}_{ji}n_ju_i$$
$$+\frac{\beta}{\gamma}\tilde{H}_in_iu_4-\frac{1}{\mathrm{i}\omega T_0}\tilde{Q}_in_iu_5)\,\mathrm{d}a = \frac{2}{\mathrm{i}\omega T_0}\int_{D^+}u_{5,i}\tilde{u}_{5,i}\,\mathrm{d}v. \tag{48}$$

5. A REPRESENTATION OF SOLUTIONS

We consider the equation

$$A\left(\frac{\partial}{\partial x}, \omega\right)U = 0. \tag{49}$$

Theorem 5.1.

Let $U = (\mathbf{u}, u_4, u_5)$ be a regular solution of eqn (49). Then

$$U = (\mathbf{u}^{(1)} + \mathbf{u}^{(2)}, u_4, u_5), \tag{50}$$

where

$$\begin{aligned}(\Delta+\sigma_1^2)(\Delta+\sigma_2^2)(\Delta+\sigma_3^2)\mathbf{u}^{(1)} &= \mathbf{0}, \qquad \operatorname{curl} \mathbf{u}^{(1)} = \mathbf{0},\\ (\Delta+\sigma_4^2)\mathbf{u}^{(2)} = \mathbf{0}, &\quad (\Delta+\sigma_1^2)(\Delta+\sigma_2^2)(\Delta+\sigma_3^2)u_4 = 0,\\ \operatorname{div} \mathbf{u}^{(2)} = 0, &\quad (\Delta+\sigma_1^2)(\Delta+\sigma_2^2)(\Delta+\sigma_3^2)u_5 = 0.\end{aligned} \tag{51}$$

Proof. Equation (49) implies that

$$\begin{aligned}\mu\Delta\mathbf{u} + (\lambda+\mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \omega^2\mathbf{u} + \beta \operatorname{grad} u_4 - b \operatorname{grad} u_5 &= \mathbf{0},\\ \Delta u_4 + \chi u_4 + m u_5 - \gamma \operatorname{div} \mathbf{u} &= 0,\\ \Delta u_5 + \mathrm{i}\omega T_0 c u_4 + \mathrm{i}\omega T_0 a u_5 + \mathrm{i}\omega T_0 b \operatorname{div} \mathbf{u} &= 0.\end{aligned} \tag{52}$$

By curl curl $\mathbf{u} =$ grad div $\mathbf{u} - \Delta\mathbf{u}$, the first equation from (52) becomes

$$(\lambda+2\mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \omega^2\mathbf{u} - \mu \operatorname{curl} \operatorname{curl} \mathbf{u} + \beta \operatorname{grad} u_4 - b \operatorname{grad} u_5 = \mathbf{0}.$$

Thus, we can write

$$\mathbf{u} = \mathbf{u}^{(1)} + \mathbf{u}^{(2)}, \tag{53}$$

where

$$\begin{aligned}\mathbf{u}^{(1)} &= -\frac{\lambda+2\mu}{\omega^2} \operatorname{grad} \operatorname{div} \mathbf{u} - \frac{\beta}{\omega^2} \operatorname{grad} u_4 + \frac{b}{\omega^2} \operatorname{grad} u_5,\\ u^{(2)} &= \frac{\mu}{\omega^2} \operatorname{curl} \operatorname{curl} \mathbf{u}.\end{aligned} \tag{54}$$

Clearly,

$$\operatorname{curl} \mathbf{u}^{(1)} = \mathbf{0}, \quad \operatorname{div} \mathbf{u}^{(2)} = 0. \tag{55}$$

From eqn (52) we get $(\mu\Delta + \omega^2) \operatorname{curl} \mathbf{u} = \mathbf{0}$, so that

$$(\Delta+\sigma_4^2)\mathbf{u}^{(2)} = \mathbf{0}. \tag{56}$$

Equations (52) imply that

$$(\Delta+p^2)\,\mathrm{div}\,\mathbf{u}+\frac{\beta}{\lambda+2\mu}\Delta u_4-\frac{b}{\lambda+2\mu}\Delta u_5=0,$$
$$(\Delta+\chi)u_4-\gamma\,\mathrm{div}\,\mathbf{u}+mu_5=0,$$
$$(\Delta+\mathrm{i}\omega T_0 a)u_5+\mathrm{i}\omega T_0 b\,\mathrm{div}\,\mathbf{u}+\mathrm{i}\omega T_0 c u_4=0, \tag{57}$$

where

$$p^2=\frac{\omega^2}{\lambda+2\mu}.$$

By eqn (57)

$$\Big\{(\Delta+p^2)[(\Delta+\chi)(\Delta+\mathrm{i}\omega T_0 a)-\mathrm{i}\omega T_0 mc]+\frac{\beta}{\lambda+2\mu}\Delta[\gamma(\Delta+\mathrm{i}\omega T_0 a)+\mathrm{i}\omega T_0 mb]$$
$$+\frac{b}{\lambda+2\mu}\mathrm{i}\omega T_0\Delta[b(\Delta+\chi)+c\gamma]\Big\}\,\mathrm{div}\,\mathbf{u}=0.$$

The above relation can be written

$$(\Delta+\sigma_1^2)(\Delta+\sigma_2^2)(\Delta+\sigma_3^2)\,\mathrm{div}\,\mathbf{u}=\mathbf{0},$$

where σ_1^2, σ_2^2, σ_3^2 are the roots of eqn (23). Similarly, we find that

$$(\Delta+\sigma_1^2)(\Delta+\sigma_2^2)(\Delta+\sigma_3^2)u_4=0,\quad (\Delta+\sigma_1^2)(\Delta+\sigma_2^2)(\Delta+\sigma_3^2)u_5=0,$$

and this completes the proof.

6. RADIATION CONDITIONS. A UNIQUENESS RESULT

In this section we study the regular solutions in D^- of eqn (49). We establish conditions at infinity of Sommerfeld type and show that these conditions ensure a unique solution.

Let $U=(\mathbf{u}^{(1)}+\mathbf{u}^{(2)},u_4,u_5)$ be a regular solution in D^- of eqn (49). We say that U satisfies the radiation conditions if

$$\mathbf{u}^{(1)}(x)=o(r^{-1}),\qquad \mathbf{u}^{(1)}_{,j}=O(r^{-2}),$$
$$u_L(x)=o(r^{-1}),\qquad u_{L,j}=O(r^{-2}),\quad (L=4,5)$$
$$\mathbf{u}^{(2)}(x)=O(r^{-1}),\qquad \frac{\partial\mathbf{u}^{(2)}}{\partial r}-\mathrm{i}\sigma_4\mathbf{u}^{(2)}=o(r^{-1}), \tag{58}$$

where $x=(x_1,x_2,x_3)$ and $r^2=x_ix_i$. Here, $f=O(g)$ means that f/g is bounded, and $f=o(g)$ means that f/g tends to zero when g increases.

We introduce the notation

$$T_i^e(\mathbf{u})=[\lambda u_{r,r}\delta_{ij}+\mu(u_{i,j}+u_{j,i})]n_j. \tag{59}$$

It is known from Kupradze (1965) that if $(58)_3$ holds then we have

$$\mathbf{u}^{(2)}(x)=O(r^{-1}),\quad \mathbf{T}^e(\mathbf{u}^{(2)})-\mathrm{i}\omega\sqrt{\mu}\,\mathbf{u}^{(2)}=o(r^{-1}). \tag{60}$$

Theorem 6.1.

Let $U = (\mathbf{u}^{(1)}+\mathbf{u}^{(2)}, u_4, u_5)$ be a regular solution in D^- of eqn (49) that satisfies the radiation conditions and one of the conditions

$$\mathbf{u} = \mathbf{0}, \quad u_4 = 0, \quad u_5 = 0, \quad \text{on } \mathscr{F}, \tag{61}$$

$$T_{ji}n_j = 0, \quad H_i n_i = 0, \quad Q_i n_i = 0, \quad \text{on } \mathscr{F}. \tag{62}$$

Then $\mathbf{u}^{(1)} = \mathbf{0}$, $\mathbf{u}^{(2)} = \mathbf{0}$, $u_4 = 0$, $u_5 = 0$.

Proof. We consider the sphere $\Sigma(0;R)$ with sufficiently large R such that $D^+ \subset \Sigma(0;R)$. Let $\partial\Sigma$ be the boundary of $\Sigma(0;R)$ and $\Omega = D^- \cap \Sigma(0;R)$. If $U = (\mathbf{u}, u_4, u_5)$ is a regular solution in D^- of eqn (49), that satisfies the radiation conditions and one of the conditions (61), (62), then by (48) we get

$$\begin{aligned}\frac{2\mathrm{i}}{\omega T_0}\int_\Omega \vartheta_{,i}\bar{\vartheta}_{,i}\,\mathrm{d}v &= \int_{\partial\Sigma}[u_i^{(2)}\bar{T}_i^{\mathrm{e}}(\mathbf{u}^{(2)}) - \bar{u}_i^{(2)}T_i^{\mathrm{e}}(\mathbf{u}^{(2)})]\,\mathrm{d}a + o(1)\\ &= \int_{\partial\Sigma}[u_i^{(2)}(\bar{T}_i^{\mathrm{e}}(\mathbf{u}^{(2)}) + \mathrm{i}\omega\sqrt{\mu}\bar{u}_i^{(2)}) - \bar{u}_i^{(2)}(T_i^{\mathrm{e}}(\mathbf{u}^{(2)}) - \mathrm{i}\omega\sqrt{\mu}u_i^{(2)})\\ &\qquad - 2\mathrm{i}\omega\sqrt{\mu}u_i^{(2)}\bar{u}_i^{(2)}]\,\mathrm{d}a + o(1).\end{aligned} \tag{63}$$

Clearly, eqn (60) implies that

$$\bar{T}^{\mathrm{e}}(\mathbf{u}^{(2)}) + \mathrm{i}\omega\sqrt{\mu}\mathbf{u}^{(2)} = o(r^{-1}). \tag{64}$$

Thus, if we let $R \to \infty$ and use eqn (64), we conclude from (63) that

$$\frac{1}{\omega T_0}\int_{D^-}|u_{5,i}|^2\,\mathrm{d}v + \omega\sqrt{\mu}\lim_{R\to\infty}\int_{\partial\Sigma}|\mathbf{u}^{(2)}|^2\,\mathrm{d}a = 0.$$

Therefore

$$\int_{D^-}|u_{5,i}|^2\,\mathrm{d}v = 0, \quad \lim_{R\to\infty}\int_{\partial\Sigma}|\mathbf{u}^{(2)}|^2\,\mathrm{d}a = 0. \tag{65}$$

In view of Theorem 5.1,

$$(\Delta + \sigma_4^2)\mathbf{u}^{(2)} = 0. \tag{66}$$

By a well known theorem [see Kupradze (1965), p. 53] from $(65)_2$ and (66) we obtain $\mathbf{u}^{(2)} = \mathbf{0}$. It follows from eqns $(65)_1$ and $(58)_2$ that $u_5 = 0$. By using eqns (52) and $(58)_2$ we find that $u_4 = 0$ and $\operatorname{div}\mathbf{u}^{(1)} = 0$. Moreover from $\operatorname{div}\mathbf{u}^{(1)} = 0$, $\operatorname{curl}\mathbf{u}^{(1)} = \mathbf{0}$ and the conditions at infinity we conclude that $\mathbf{u}^{(1)} = \mathbf{0}$, and the proof is complete.

7. SOMIGLIANA RELATIONS. POTENTIALS

Following Kupradze (1965) and Kupradze *et al.* (1979), if $U = (u_1, u_2, u_3, u_4, u_5)$ is a regular vector field in D^+ and $V = \tilde{\Gamma}^{(k)}(x, y; \omega)$, $(k = 1, 2, \ldots, 5)$, $y \in D^+$, then eqn (39) implies that

$$U_k(y) = \int_{\mathscr{F}} \left[U(x)\tilde{R}\left(\frac{\partial}{\partial x}, n_x\right)\tilde{\Gamma}^{(k)}(x,y;\omega) - \tilde{\Gamma}^{(k)}(x,y;\omega)R\left(\frac{\partial}{\partial x}, n_x\right)U(x)\right] \mathrm{d}a_x$$

$$+ \int_{D^+} \tilde{\Gamma}^{(k)}(x,y;\omega)A\left(\frac{\partial}{\partial x}, n_x\right)U(x)\,\mathrm{d}v_x, \quad (k = 1,2,\ldots,5). \quad (67)$$

Taking into account eqn (27) from (67) we obtain

$$U(x) = \int_{D^+} \Gamma(x,y;\omega)A\left(\frac{\partial}{\partial y}, \omega\right)U(y)\Big]\,\mathrm{d}v_y$$

$$+ \int_{\mathscr{F}} \left[\Lambda(x,y;\omega)U(y) - \Gamma(x,y;\omega)R\left(\frac{\partial}{\partial y}, n_y\right)U(y)\right]\mathrm{d}a_y, \quad x \in D^+, \quad (68)$$

where

$$\Lambda(x,y;\omega) = \left[\tilde{R}\left(\frac{\partial}{\partial y}, n_y\right)\Gamma^*(x,y;\omega)\right]^*. \quad (69)$$

The following result can be established as in Kupradze (1965) and Kupradze *et al.* (1979).

Theorem 7.1.

Each column of the matrix $\Lambda(x,y;\omega)$ satisfies at x the eqn (49), for $x \neq y$.

If U is a regular vector field in D^- that satisfies the radiation conditions, then we can establish a representation of U analogous with eqn (68).

Following Kupradze (1965) and Kupradze *et al.* (1979), we introduce the potential of a single-layer

$$V(x;\psi) = \int_{\mathscr{F}} \Gamma(x,y;\omega)\psi(y)\,\mathrm{d}a_y, \quad (70)$$

the potential of a double-layer

$$W(x;\eta) = \int_{\mathscr{F}} \Lambda(x,y;\omega)\eta(y)\,\mathrm{d}a_y, \quad (71)$$

and the potential of mass

$$U(x;\rho) = \int_{D^+} \Gamma(x,y;\omega)\rho(y)\,\mathrm{d}v_y. \quad (72)$$

We assume that ψ is Hölder continuous on $\mathscr{F}$, η is Hölder continuously differentiable on $\mathscr{F}$ and ρ is Hölder continuous on $\bar{D}^+$. As in the classical thermoelasticity (Kupradze *et al.*, 1979), we have

$$A\left(\frac{\partial}{\partial y}, \omega\right)V(x;\psi) = 0, \quad A\left(\frac{\partial}{\partial y}, \omega\right)W(x;\eta) = 0, \quad A\left(\frac{\partial}{\partial x}, \omega\right)U(x;\rho) = \rho, \quad x \in D^+. \quad (73)$$

Moreover, following Kupradze (1965) and Kupradze *et al.* (1979), we can prove the following theorems.

Theorem 7.2.

The potential of the single-layer is continuous throughout.

Theorem 7.3.

The potential of a double-layer has finite limits when the point x tends to $z \in \mathscr{F}$ both from within and from without, and these limits are respectively equal to

$$W^{+}(z;\eta) = -\tfrac{1}{2}\eta(z) + \int_{\mathscr{F}} \Lambda(z,y;\omega)\eta(y)\,\mathrm{d}a_y,$$

$$W^{-}(z;\eta) = \tfrac{1}{2}\eta(z) + \int_{\mathscr{F}} \Lambda(z,y;\omega)\eta(y)\,\mathrm{d}a_y, \tag{74}$$

the integrals being conceived in the sense of Chauchy's principal value.

Theorem 7.4.

$R(\partial/\partial x, n_x)V(x;\psi)$ has finite limits as the point x tends to $z \in \mathscr{F}$ from within and from without and these limits are

$$\left[R\left(\frac{\partial}{\partial z}, n_z\right)V(z;\psi)\right]^{+} = \tfrac{1}{2}\psi(z) + \int_{\mathscr{F}} \left[R\left(\frac{\partial}{\partial z}, n_z\right)\Gamma(z,y;\omega)\right]\psi(y)\,\mathrm{d}a_y,$$

$$\left[R\left(\frac{\partial}{\partial z}, n_z\right)V(z;\psi)\right]^{-} = -\tfrac{1}{2}\psi(z) + \int_{\mathscr{F}} \left[R\left(\frac{\partial}{\partial z}, n_z\right)\Gamma(z,y;\omega)\right]\psi(y)\,\mathrm{d}a_y, \tag{75}$$

respectively.

In what follows we restrict our attention to eqn (49). We consider the following boundary problems.

Interior problems

To find a regular solution in D^{+} of eqn (49) satisfying one of the conditions

$$U^{+} = G_1, \quad \text{on } \mathscr{F}, \tag{I$_1$}$$

$$\left[R\left(\frac{\partial}{\partial z}, n_z\right)U(z)\right]^{+} = G_2(z), \quad z \in \mathscr{F}, \tag{I$_2$}$$

where G_1 and G_2 are prescribed vector fields.

Exterior problems

To find a regular solution in D^{-} of eqn (49) that satisfies the radiation conditions and one of the conditions

$$U^{-} = G_3, \quad \text{on } \mathscr{F}, \tag{E$_1$}$$

$$\left[R\left(\frac{\partial}{\partial z}, n_z\right)U(z)\right]^{-} = G_4(z), \quad z \in \mathscr{F}, \tag{E$_2$}$$

where G_3 and G_4 are prescribed vector fields.

We assume that G_1 and G_3 are Hölder continuously differentiable on $\mathscr{F}$, and G_2 and G_4 are Hölder continuous on $\mathscr{F}$. We denote by (I_α^0) and (E_α^0) the homogeneous problems corresponding to (I_α) and (E_α), $(\alpha = 1, 2)$, respectively.

We seek the solutions of the problems (I_1) and (E_1) in the form of the double-layer potential and the solutions of the problems (I_2) and (E_2) in the form of the single-layer potential. In view of Theorems 7.3 and 7.4 we obtain for unknown densities the following equations

$$-\tfrac{1}{2}\eta(z)+\int_{\mathscr{F}}\Lambda(z,y;\omega)\eta(y)\,\mathrm{d}a_y = G_1(z), \qquad (\mathrm{I}_1)$$

$$\tfrac{1}{2}\psi(z)+\int_{\mathscr{F}}\left[R\left(\frac{\partial}{\partial z},n_z\right)\Gamma(z,y;\omega)\right]\psi(y)\,\mathrm{d}a_y = G_2(z), \qquad (\mathrm{I}_2)$$

$$\tfrac{1}{2}\eta(z)+\int_{\mathscr{F}}\Lambda(z,y;\omega)\eta(y)\,\mathrm{d}a_y = G_3(z), \qquad (\mathrm{E}_1)$$

$$-\tfrac{1}{2}\psi(z)+\int_{\mathscr{F}}\left[R\left(\frac{\partial}{\partial z},n_z\right)\Gamma(z,y;\omega)\right]\psi(y)\,\mathrm{d}a_y = G_4(z), \qquad (\mathrm{E}_2)$$

$z\in\mathscr{F}$. These equations are two-dimensional singular integral equations; they can only be understood in the sense of principal values. Let us show that the Fredholm theory is applicable. The integral operator of the problem (I_1) may be written in the form

$$H\eta = -\eta(z)+\int_{\mathscr{F}}K(z,y;\omega)\eta(y)\,\mathrm{d}a_y, \qquad (76)$$

where

$$K(z,y) = 2\Lambda(z,y;\omega).$$

It follows from eqns (27) and (69) that

$$K(z,y) = K_1(z,y)+K_2(z,y),$$

where

$$K_1(z,y) = \|k_{rs}(z,y)\|_{5\times 5},$$

$$k_{rs}(z,y) = \frac{\mu}{2\pi(\lambda+2\mu)r^3}[(z_s-y_s)\cos(n_y,z_r)-(z_r-y_r)\cos(n_y,z_s)],$$

$$k_{4j} = k_{j4} = k_{44} = k_{5j} = k_{j5} = k_{45} = k_{54} = k_{55} = 0,$$

$$K_2(z,y) = O(|z-y|^{\alpha-2}), \quad \alpha \geqslant 0.$$

Next, we determine the characteristic matrix of the operator H introducing a local coordinate system $(\zeta_1,\zeta_2,\zeta_3)$ at each point $z\in\mathscr{F}$ with the ζ_3-axis directed along the outside normal to $\mathscr{F}$ at z. Let $k=\|k_{ij}\|_{5\times 5}$ be the characteristic matrix of H and $\sigma=\|\sigma_{ij}\|_{5\times 5}$ be the symbolic matrix of the operator H. We find that

$$k_{11}=k_{22}=k_{33}=k_{44}=k_{55}=-1, \quad k_{ij}=-k_{ji}, \quad (i\neq j),$$

$$k_{i4}=k_{i5}=k_{4i}=k_{5i}=0, \quad \sigma_{11}=\sigma_{22}=\sigma_{33}=\sigma_{44}=\sigma_{55}=-1,$$

$$\sigma_{rs}=2\pi \mathrm{i}k_{rs}\ (r\neq s), \quad \sigma_{i4}=\sigma_{i5}=\sigma_{4i}=\sigma_{5i}=0,$$

and

$$\det \sigma = 1 - \frac{\mu^2}{(\lambda+2\mu)^2}.$$

In what follows we assume that

$$\frac{\mu}{(\lambda+2\mu)} \neq \pm 1,$$

so that the integral operator corresponding to the equation (I_1) is of the normal type. Since H is of normal type and the symbolic matrix is antisymmetric, then the index of this operator is zero (cf. Mikhlin, 1965).

Thus, we may conclude that the operator H is a Fredholm operator. In a similar way we can prove that the operators corresponding to the remaining equations are Fredholm operators in the space of Hölder continuous functions.

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